A recipe to solve (some) Stochastic Differential Equations analytically

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I recently delved into the fascinating world of stochastic differential equations (in short SDEs) through an open MIT course [1]. Amidst the richness of intriguing concepts, one particular point caught my attention and left me pondering. I felt like sharing this point with you.

The lecturer presented a recipe for solving SDEs analytically. They however said that this recipe would only work for a few SDEs, that in other cases it would fail and that instead you may have to feel the solution and verify it by plugging it into the SDE. They then went on saying that anyways, nowadays people would use computers to solve these equations.

If you share my enthusiasm for analytic solutions and have a desire to solve SDEs - even in a scenario where you might choose to reside in a secluded forest cabin devoid of electricity - then relying on a computer feels unsatisfactory.

While I am an admirer of verifying magically guessed solutions, I tend to forget relevant details of these it-makes-you-look-like-a-genius approaches.

So, I was thinking, can we extend the recipe so that it is useful for more SDEs, and in a way that is easy to memorize?

The lecturer considered SDEs of the following form

$$\begin{aligned} dX_t &= \mu(X_t, t)dt + \sigma(X_t, t)dB_t \\ X_0 &= x \end{aligned}$$

where B is a Brownian motion, μ and σ are sufficiently nice functions to guarantee existence and uniqueness of a solution, and x is a real number.

An (explicit) analytic solution is an expression $X_t = \ldots$ with no X-terms on its right side. The presented recipe was to assume

$$X_t = f(B_t, t) \tag{1}$$

for a function f to be determined as follows. First, apply Itô's formula to obtain

$$df = \left(f_t + \frac{1}{2} f_{BB}\right) dt + f_B dB_t$$

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where f_t, f_B and f_{BB} denote partial derivatives. Second, match the terms in this expression with the SDE to find

$$\begin{aligned} dt: \quad & \mu(f,t) = \left(f_t + \frac{1}{2} f_{BB}\right) \\ dB_t: \quad & \sigma(f,t) = f_B. \end{aligned}$$

Third and final, find the function f which solves these equations.

While this recipe works to solve the SDE of the geometric Brownian motion (meaning $\mu(X_t,t) = \mu X_t$ and $\sigma(X_t,t) = \sigma X_t$), it fails for the Ornstein-Uhlenbeck (OU) SDE (meaning $\mu(X_t,t) = -\mu X_t$ and $\sigma(X_t,t) = \sigma$). There is a well known method to neatly solve the OU SDE (using an integrating factor), but let's try a different approach, an extended recipe, and see if it works for other SDEs as well.

What I am considering is to include two additional variables in (1):

$$\boxed{X_t = f(B_t, t, \underbrace{\int_0^t g(s) dB_s}_{=y_t}, \underbrace{\int_0^t h(B_s, s) ds}_{=z_t})} \tag{2}$$

Let's refer to these additional variables, the integrals, as y and z. Hence, to solve the SDE we have to find f, g and h. This is done by matching the derivatives of f with their counterparts in the SDE, as was done for the presented approach (1) in the MIT course:

$$dt: \quad \mu(f,t) = \left(f_t + \frac{1}{2} f_{BB}\right) + f_z h(B_t, t) \tag{3}$$

$$dB_t: \quad \sigma(f,t) = f_B + f_y g(t). \tag{4}$$

This extended recipe will continue to require some guessing, but it should aid in making meaningful conjectures. Indeed, finding a solution should feel more like a walk through a maze where the turns correspond to trying meaningful f_t, f_B, f_{BB}, $f_z h()$ and $f_y g()$.

Let's apply this modified recipe!

In all following examples, the initial condition is $X_0 = x > 0$. Further, all stochastic integrals are understood as Itô integrals.

Example 1) The Ornstein-Uhlenbeck (OU) process with drift

The SDE to solve is:

$$dX_t = \theta(\mu - X_t) dt + \sigma dB_t.$$

Writing $X_t = f$, then (3) and (4) become:

$$\begin{aligned} dt: \quad & \theta(\mu - f) = \left(f_t + \frac{1}{2} f_{BB}\right) + f_z h(B_t, t) \\ dB_t: \quad & \sigma = f_B + f_y g(t). \end{aligned}$$

The second equation suggests to assume that f_B and $f_y g(t)$ are constants (because the left side is constant). f_B being constant implies $f_{BB} = 0$. If we additionally presume that $f_z = 0$, the first equation simplifies to an ordinary differential equation (ODE). Trying $f_z = 0$ corresponds to a turn in the maze which seems meaningful because the resulting ODE is sufficient to obtain a correct match with the SDE's dt-term. With standard techniques (details can be found in the appendix) we find that the solution of this ODE is:

$$f = Ce^{-\theta t} + \mu.$$

We however need to consider that C may be a function of B and y, meaning $C(B_t, y_t)$ (excluding z because we assumed $f_z = 0$), and that $C(0,0) = x - \mu$ to meet the initial condition (because B_0 and y_0 are both zero). Hence, the solution of the ODE takes the form:

$$f = (k(B_t, y_t) + x - \mu)e^{-\theta t} + \mu$$

where k is a function to be determined.

We know that this function k must be linear in B (because otherwise $f_{BB} = 0$ would be violated), and that it must satisfy $k(0,0) = 0$. We finally use the dB_t-equation to find that $k = y$ and $g(t) = \sigma e^{\theta t}$ are feasible.

This reveals the solution of the SDE:

$$\boxed{f = X_t = \sigma \int_0^t e^{-\theta(t-s)} dB_s + e^{-\theta t}(x - \mu) + \mu}$$

Example 2) A square root process

The SDE to solve is:

$$dX_t = dt + 2\sqrt{X_t}dB_t.$$

Writing $X_t = f$, then (3) and (4) become:

$$\begin{aligned} dt: \quad & 1 = \left(f_t + \frac{1}{2} f_{BB}\right) + f_z h(B_t, t) \\ dB_t: \quad & 2\sqrt{f} = f_B + f_y g(t). \end{aligned}$$

The second equation suggests to try $f_y = 0$ and a quadratic function in B,

$$\boxed{f = X_t = (x + B_t)^2}$$

Since this also solves the dt-equation for $f_z = 0$, we have already found the solution of the SDE.

Example 3) A geometric Brownian motion with a drift and a time trend

The SDE to solve is:

$$dX_t = (\mu X_t + \alpha)dt + \sigma X_t dB_t. \tag{5}$$

Writing $X_t = f$, then (3) and (4) become:

$$\begin{aligned} dt: &\quad \mu f + \alpha = \left(f_t + \frac{1}{2} f_{BB} \right) + f_z h(B_t, t) \\ dB_t: &\quad \sigma f = f_B + f_y g(t). \end{aligned}$$

If α were zero, then the solution would be the standard geometric Brownian motion,

$$X_t = x e^{(\mu - \sigma^2/2)t + \sigma B_t}.$$

This hints to try $f_y = 0$ and use f_z to match the αdt-term.

Given that the time trend αdt enters the SDE additively, and recalling the derivative rule for products ($(fg)' = f'g + fg'$) we proceed with:

$$f = x e^{(\mu - \sigma^2/2)t + \sigma B_t} k(z_t) \tag{6}$$

for a function k to be determined. Recall that by definition z is a function of t,

$$z_t = \int_0^t h(B_s, s) ds,$$

with derivative $h(B_t, t)$. Therefore, the dt-equation becomes (using the product rule and the chain rule):

$$dt: \quad \mu f + \alpha = \mu f + x e^{(\mu - \sigma^2/2)t + \sigma B_t} k'(z_t) h(B_t, t).$$

Hence,

$$\alpha = x e^{(\mu - \sigma^2/2)t + \sigma B_t} k'(z_t) h(B_t, t)$$

which determines k to be of the form

$$k(z) = k_1 z + k_2 = k_1 \int_0^t h(B_s, s) ds + k_2 \tag{7}$$

with

$$\begin{aligned} k_1 &= \frac{\alpha}{x} \\ k_2 &= 1 \\ h(B_t, t) &= e^{-(\mu - \sigma^2/2)t - \sigma B_t}. \end{aligned}$$

Combining (6) and (7) gives the solution of the SDE:

$$\boxed{f = X_t = e^{(\mu - \sigma^2/2)t + \sigma B_t} \left(\alpha \int_0^t e^{-(\mu - \sigma^2/2)s - \sigma B_s} \, ds + x \right)}$$

Example 4) A geometric Brownian motion with an inverse drift

The SDE to solve is:

$$dX_t = \frac{1}{X_t} dt + \sigma X_t dB_t.$$

Multiplying both sides by X_t hints at a variable substitution:

$$Y_t = X_t^2/2.$$

Y_t satisfies the SDE:

$$dY_t = X_t dX_t + \frac{1}{2} d\langle X_t, X_t \rangle$$

where $\langle X_t, X_t \rangle$ is the quadratic variation of X_t (this follows from Itô's change-of-variable formula, see e.g. [2]). The SDE for X_t defines this quadratic variation as $\sigma^2 X_t^2$. Therefore,

$$\begin{aligned} dY_t &= 1dt + \sigma X_t^2 dB_t + \frac{1}{2}\sigma^2 X_t^2 dt \\ &= (\sigma^2 Y_t + 1)dt + 2\sigma Y_t dB_t. \end{aligned}$$

This SDE for Y_t is of the form (5) which we just solved in the previous example. Hence,

$$Y_t = e^{(\sigma^2 - 2\sigma^2)t + 2\sigma B_t} \left(\int_0^t e^{-(\sigma^2 - 2\sigma^2)s - 2\sigma B_s} \, ds + \frac{x^2}{2} \right).$$

Simplifying and substituting back for X_t provides the solution of the SDE:

$$\boxed{X_t = e^{-(\sigma^2/2)t + \sigma B_t} \sqrt{\left(2 \int_0^t e^{\sigma^2 s - 2\sigma B_s} \, ds + x^2 \right)}}$$

I hope that you enjoyed this article. Let me know if you could apply this recipe and whether you found it useful. Let me also know if someone else had already come up with a similar or the same approach before me, or if you know a better way to solve SDEs analytically, and, of course, if you noticed any error.

APPENDIX

The ODE in the OU-SDE

We have to solve

$$dt : \theta(\mu - f) = f_t$$

The homogeneous and particular solutions of this ODE are

$$\begin{aligned} f^h &= Ce^{-\theta t} \\ f^p &= \mu \end{aligned}$$

Hence,

$$f = f^h + f^p = Ce^{-\theta t} + \mu.$$

The initial condition $X_0 = x$ determines that at $t = 0$

$$C + \mu = x.$$

So far, this has been a standard ODE solving procedure. Now, what is non-standard is that C may be a function of B and y, $C(B, y)$, and that we need $C(0, 0) = x - \mu$ to comply with the initial condition (because B_0 and y_0 are both zero). Hence, this was the reason for defining

$$C(B, y) = k(B, y) + x - \mu.$$

References

[1] MIT, Instructor: Dr. Choongbum Lee, Fall 2013 *Lecture 21: Stochastic Differential Equations.* https://ocw.mit.edu/courses/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/resources/lecture-21-stochastic-differential-equations/

[2] Peskir, G. and Shiryaev, A. 2006 *Optimal Stopping and Free-Boundary Problems.* Birkhäuser Verlag.